

# SEPARATORS OF FAT POINTS IN $\mathbb{P}^n \times \mathbb{P}^m$

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**ABSTRACT.** We introduce definitions for the separator of a fat point and the degree of a fat point for a fat point scheme  $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ , and we study some of their properties.

## 1. INTRODUCTION

A separator of a point and its degree are two tools in the toolbox used to study points in projective space. Recall that if  $X \subseteq \mathbb{P}^n$  is a finite set of points, and  $P \in X$ , then a **separator** of  $P$  is any homogeneous form  $F \in R = k[\mathbb{P}^n] = k[x_0, \dots, x_n]$  such that  $F(P) \neq 0$ , but  $F(Q) = 0$  for all  $Q \in X \setminus \{P\}$ . Geometrically, a separator is a hypersurface that passes through all the points of  $X$  except  $P$ . The **degree** of the point  $P$ , denoted  $\deg_X(P)$ , is then the smallest degree of any separator of  $P$ . The properties of separators and their degrees were studied by [1, 2, 3, 5, 14, 16, 18], among others.

The above cited articles focused predominately on the case of reduced sets of points in  $\mathbb{P}^n$ . There are two natural ways to generalize this work. The first such way is to consider separators of points in a multiprojective space  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ , as was the focus of [11, 12, 15]. The second way is to consider separators of more arbitrary zero-dimensional schemes in  $\mathbb{P}^n$ ; the papers [9, 14] take this point of view. In this paper, we consider the marriage of these two ideas by studying separators of non-reduced points (specifically, fat points) in a multiprojective space.

We restrict ourselves in this paper primarily to the bigraded case of  $\mathbb{P}^n \times \mathbb{P}^m$ . This restriction has the benefit of simplifying our notation when compared to the general multigraded situation, and at the same time, our results are much stronger in this context. Once we recall the required background in Section 2, we introduce in Section 3 our definition of a separator for a fat point in  $\mathbb{P}^n \times \mathbb{P}^m$ . Our approach is similar to that of [9] in that our definitions are defined in terms of the bigraded generators of the ideal  $I_{Z'}/I_Z$  in  $R/I_Z$ , where  $Z' \subseteq Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$  are fat point schemes and  $R = k[\mathbb{P}^n \times \mathbb{P}^m]$ . In Section 4 we introduce the notion of a good set of minimal separators. Roughly speaking, a good set of minimal separators allows us to describe a basis for the vector space  $(I_{Z'}/I_Z)_{\underline{t}}$  for all  $\underline{t} \in \mathbb{N}^2$ . The main results of this paper are Theorem 5.1 and Theorem 6.4. The first theorem shows that arithmetically Cohen-Macaulay (ACM) sets of fat points in  $\mathbb{P}^n \times \mathbb{P}^m$  have a good set of minimal separators. The second shows that if  $R/I_Z$  is Cohen-Macaulay (CM), the degree of a separator of a fat point is encoded into the shifts of the last syzygy module of  $I_Z$ , generalizing similar results of [1, 2, 9, 12].

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We wish to point out that although some facts for fat points in multiprojective spaces follow without any difficulty from the methods used in [9], our main results require additional development beyond what is done in [9]. This is the case because when we move to the case of (non)reduced points  $Z$  in a multiprojective spaces, we are no longer guaranteed that the associated coordinate ring  $R/I_Z$  is CM, and furthermore, even if  $R/I_Z$  is CM, it may not be true that  $R/I_{Z'}$  is CM for subschemes  $Z' \subseteq Z$ . The fact that  $R/I_Z$  and  $R/I_{Z'}$  may fail to be CM is an obstruction to generalizing some of the proofs in [9] and at the same time, highlights the importance of the CM property of zero-dimensional schemes in  $\mathbb{P}^n$ .

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## 2. PRELIMINARIES

We recall the relevant properties of points in  $\mathbb{P}^n \times \mathbb{P}^m$ . The study of such points was initiated in [6, 7]; further properties were developed in [8, 10, 11, 12, 19, 20]. Throughout,  $k$  denotes an algebraically closed field of characteristic zero.

We shall write  $(i_1, i_2) \in \mathbb{N}^2$  as  $\underline{i}$ . We induce a partial order on  $\mathbb{N}^2$  by setting  $(i_1, i_2) \succeq (j_1, j_2)$  if  $i_t \geq j_t$  for  $t = 1, 2$ . The coordinate ring of the **biprojective space**  $\mathbb{P}^n \times \mathbb{P}^m$  is the  $\mathbb{N}^2$ -graded ring  $R = k[x_0, \dots, x_n, y_0, \dots, y_m]$  where  $\deg x_i = (1, 0)$  and  $\deg y_i = (0, 1)$ . A point in this space has the form

$$P = [a_0 : \dots : a_n] \times [b_0 : \dots : b_m] \in \mathbb{P}^n \times \mathbb{P}^m$$

and its defining ideal  $I_P$  in  $R$  is a prime ideal of the form

$$I_P = (L_1, \dots, L_n, L'_1, \dots, L'_m)$$

where  $\deg L_i = (1, 0)$  and  $\deg L'_i = (0, 1)$ . When  $X = \{P_1, \dots, P_s\}$  is a set of  $s$  distinct points in  $\mathbb{P}^n \times \mathbb{P}^m$ , and  $m_1, \dots, m_s$  are positive integers, then  $I_Z = I_{P_1}^{m_1} \cap \dots \cap I_{P_s}^{m_s}$  defines a **fat point scheme** (or a **set of fat points**) which we denote by  $Z = m_1 P_1 + \dots + m_s P_s$ . We call  $m_i$  the **multiplicity** of the point  $m_i$ , and the set  $X$ , sometimes denoted by  $\text{Supp}(Z)$ , is the **support** of  $Z$ . The **degree** of a scheme of fat points  $Z = m_1 P_1 + \dots + m_s P_s$  is then given by  $\deg Z = \sum_{i=1}^s \binom{m_i + N - 1}{m_i - 1}$  where  $N = n + m$ .

The ring  $R/I_Z$  has Krull dimension 2, but  $1 \leq \text{depth } R/I_Z \leq 2$  (see [19]). When  $\dim R/I_Z = 2 = \text{depth } R/I_Z$ , we say  $Z$  is **arithmetically Cohen-Macaulay** (ACM).

We need some results about the nonzero-divisors and longest regular sequence in  $R/I_Z$ .

**Lemma 2.1.** *Let  $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$  be a set of fat points.*

- (i) *There exist two forms  $L_1$  and  $L'_1$  such that  $\deg L_1 = (1, 0)$  and  $\deg L'_1 = (0, 1)$  and both  $\overline{L_1}$ , and  $\overline{L'_1}$  are nonzero-divisors on  $R/I_Z$ .*
- (ii) *If  $Z$  is also ACM, then there exist elements  $\overline{L_1}, \overline{L'_1}$  in  $R/I_Z$  such that  $L_1, L'_1$  give rise to a regular sequence in  $R/I_Z$  and  $\deg L_1 = (1, 0)$  and  $\deg L'_1 = (0, 1)$ .*

*Proof.* Statement (i) is [20, Lemma 3.3] extended to the nonreduced case. For (ii), adapt the proof of [19, Proposition 3.2].  $\square$

**Remark 2.2.** After a change of coordinates, we can assume  $L_1 = x_0$  and  $L'_1 = y_0$  in Lemma 2.1. Thus, when  $Z$  is ACM,  $\{x_0, y_0\}$  (or  $\{y_0, x_0\}$ ) is the regular sequence on  $R/I_Z$ . This also implies that  $x_0$  and  $y_0$  do not vanish at any point of  $\text{Supp}(Z)$ .

We require a lemma about the bigraded resolution of a single point. Since  $I_P$  is a complete intersection, the proof is an application of the bigraded Koszul resolution.

**Lemma 2.3.** *Let  $P \in \mathbb{P}^n \times \mathbb{P}^m$  be any point. Then the minimal  $\mathbb{N}^2$ -graded free resolution of  $R/I_P$  has the form*

$$0 \rightarrow \mathbb{G}_N \rightarrow \mathbb{G}_{N-1} \rightarrow \cdots \rightarrow \mathbb{G}_1 \rightarrow R \rightarrow R/I_P \rightarrow 0$$

where  $N = n + m$ ,  $\mathbb{G}_N = R(-n, -m)$  and  $\mathbb{G}_{N-1} = R^n(-n+1, -m) \oplus R^m(-n, -m+1)$ .

### 3. DEFINING SEPARATORS OF FAT POINTS

We introduce the definitions of a separator and its degree for fat points in  $\mathbb{P}^n \times \mathbb{P}^m$ . The main idea is to reduce the multiplicity of a fat point by one, and then use an ideal that captures the information about passing from the larger scheme to the smaller one.

The following convention is used to simplify our hypotheses throughout the paper.

**Convention 3.1.** Consider the fat point scheme

$$Z := m_1 P_1 + \cdots + m_i P_i + \cdots + m_s P_s \subseteq \mathbb{P}^n \times \mathbb{P}^m,$$

and fix a point  $P_i \in \text{Supp}(Z)$ . We then let

$$Z' := m_1 P_1 + \cdots + (m_i - 1) P_i + \cdots + m_s P_s,$$

denote the fat point scheme obtained by reducing the multiplicity of  $P_i$  by one. If  $m_i = 1$ , then the point  $P_i$  does not appear in the support of  $Z'$ .

A separator is now defined in terms of forms that pass through  $Z'$  but not  $Z$ .

**Definition 3.2.** Let  $Z = m_1 P_1 + \cdots + m_i P_i + \cdots + m_s P_s$  be a set of fat points in  $\mathbb{P}^n \times \mathbb{P}^m$ . We say that  $F$  is a **separator of the point  $P_i$  of multiplicity  $m_i$**  if  $F \in I_{P_i}^{m_i-1} \setminus I_{P_i}^{m_i}$  and  $F \in I_{P_j}^{m_j}$  for all  $j \neq i$ .

When  $m_i = 1$  for all  $i$ , then  $Z$  is a reduced set of points, and we recover the definition studied in [11, 12, 15]. Using the notation of Convention 3.1, a form  $F$  is a separator of the point  $P_i$  of multiplicity  $m_i$  if  $F \in I_{Z'} \setminus I_Z$ . We can algebraically compare  $Z$  and  $Z'$  by studying the ideal  $I_{Z'}/I_Z$  in the ring  $R/I_Z$ . We recall a simple fact about this ideal.

**Lemma 3.3.** *Let  $Z$  and  $Z'$  be as in Convention 3.1. Then there exists  $p$  bihomogeneous polynomials  $\{F_1, \dots, F_p\}$ , where each  $F_i$  is a separator of  $P_i$  of multiplicity  $m_i$ , such that in the ring  $R/I_Z$ , the ideal  $I_{Z'}/I_Z = (\overline{F}_1, \dots, \overline{F}_p)$ . Here,  $\overline{F}_i$  denotes the class of  $F_i$ .*

*Proof.* Because  $R/I_Z$  is Noetherian, the ideal  $I_{Z'}/I_Z$  is finitely generated. If  $\{\overline{F}_1, \dots, \overline{F}_p\}$  is a set of generators, then each  $F_i \in I_{Z'} \setminus I_Z$ .  $\square$

**Definition 3.4.** We call the set of bihomogeneous forms  $\{F_1, \dots, F_p\} \subseteq R$  a **set of minimal separators of  $P_i$  of multiplicity  $m_i$**  if

- (a)  $I_{Z'}/I_Z = (\overline{F}_1, \dots, \overline{F}_p)$ , and
- (b) there does not exist a set  $\{G_1, \dots, G_q\}$  with  $q < p$  such that  $I_{Z'}/I_Z = (\overline{G}_1, \dots, \overline{G}_q)$ .

**Remark 3.5.** Our approach is similar to [14] in that we relate a separator to generators of an ideal of a smaller scheme modulo an ideal of a larger scheme. The focus of [14] was primarily on the case that  $X$  is a zero-dimensional scheme, and  $Y \subseteq X$  is a subscheme with  $\deg Y = \deg X - 1$ . Rather than an arbitrary zero-dimensional scheme, we are interested in fat point schemes  $Z' \subseteq Z$  which normally have  $\deg Z' < \deg Z - 1$ .

Our next step is to develop a fat point analog for the degree of a point.

**Theorem 3.6.** *Let  $Z$  and  $Z'$  be as in Convention 3.1, and fix a total ordering  $\leq$  of  $\mathbb{N}^2$ . Let  $\{F_1, \dots, F_p\}$  and  $\{G_1, \dots, G_p\}$  be two sets of minimal separators of  $P_i$  of multiplicity  $m_i$ . Relabel the  $F_i$ 's so that  $\deg F_1 \leq \dots \leq \deg F_p$ , and similarly for the  $G_i$ 's. Then*

$$(\deg F_1, \dots, \deg F_p) = (\deg G_1, \dots, \deg G_p).$$

*Proof.* Let  $W = (I_{Z'}/I_Z)$ . Both  $\{\overline{F}_1, \dots, \overline{F}_p\}$  and  $\{\overline{G}_1, \dots, \overline{G}_p\}$  are a minimal set of generators for this ideal. The number of generators of degree  $\underline{d}$  of  $W$  is the dimension of

$$Y = W_{\underline{d}} / (R_{e_1} W_{\underline{d}-e_1} + R_{e_2} W_{\underline{d}-e_2})$$

as a vector space. Here,  $W_{\underline{j}}$  is the vector space of all the forms of degree  $\underline{j}$  in  $W$ ,  $R_{e_i}$  denotes the elements of degree  $e_i$  in  $R$ , and  $R_{e_i} W_{\underline{d}-e_i} = \{V_1 V_2 \mid V_1 \in R_{e_i} \text{ and } V_2 \in W_{\underline{d}-e_i}\}$ . The generators of degree  $\underline{d}$  in  $\{\overline{F}_1, \dots, \overline{F}_p\}$  and  $\{\overline{G}_1, \dots, \overline{G}_p\}$  therefore form a basis for  $Y$ , thus implying that the number of generators of degree  $\underline{d}$  is the same.  $\square$

In light of Theorem 3.6, we can define the degree of a fat point.

**Definition 3.7.** Let  $\{F_1, \dots, F_p\}$  be any set of minimal separators of  $P_i$  of multiplicity  $m_i$ , and relabel so that  $\deg F_1 \leq \dots \leq \deg F_p$  with respect to any total ordering on  $\mathbb{N}^2$ . Then the **degree of the minimal separators of  $P_i$  of multiplicity  $m_i$**  is

$$\deg_Z(P_i) := (\deg F_1, \dots, \deg F_p) \text{ with } \deg F_i \in \mathbb{N}^2.$$

We illustrate some of the above ideas with the following two examples.

**Example 3.8.** Let  $Z = mP$  be a single fat point of multiplicity  $m \geq 2$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ . We can assume that  $I_P = (x_1, y_1)$ , and hence  $I_Z = I_P^m$ . Then

$$I_{Z'}/I_Z = I_P^{m-1}/I_P^m = (\overline{M} \mid M = x_1^a y_1^b \text{ with } a + b = m - 1).$$

The generators of  $I_P^{m-1}$  are a set of minimal separators of  $P$  of multiplicity  $m$ , whence

$$\deg_Z(P) = ((0, m-1), (1, m-2), \dots, (m-2, 1), (m-1, 0)).$$

Note that in this case we have  $m = |\deg_Z(P)| = \deg Z - \deg Z'$ . The situation where  $|\deg_Z(P)| = \deg Z - \deg Z'$  plays an important role in the next section.

**Example 3.9.** We consider two fat points  $Z = 2P_1 + 2P_2$  where  $P_1 = [1 : 0 : 0] \times [1 : 0 : 0 : 0]$  and  $P_2 = [0 : 0 : 1] \times [0 : 0 : 0 : 1]$  in  $\mathbb{P}^2 \times \mathbb{P}^3$ . Note that  $I_Z$  is monomial ideal since  $I_{P_1}$  and  $I_{P_2}$  are monomial ideals.

Let  $Z' = 2P_1 + P_2$ . To find the separators of  $P_2$  of multiplicity 2, it is enough to determine which generators of  $I_{Z'}$  do not belong to  $I_Z$ . Using CoCoA [4], we get

$$\{x_1x_2, x_1y_3, x_2y_1, x_2y_2, y_1y_3, y_2y_3, x_0x_2^2, x_2^2y_0, x_0x_2y_3, x_2y_0y_3, x_0y_3^2, y_0y_3^2\}.$$

It then follows that

$$\deg_Z(P_2) = ((0, 2), (0, 2), (0, 3), (1, 1), (1, 1), (1, 1), (1, 2), (1, 2), (2, 0), (2, 1), (2, 1), (3, 0)),$$

where we ordered our tuples with respect to the lex ordering. Note that  $|\deg_Z(P_2)| = 12$ , which does not equal  $\deg Z - \deg Z' = 5$ . In this case,  $Z$  is not ACM.

#### 4. GOOD SEPARATORS

We introduce the notion of a good set of minimal separators. Roughly speaking, a minimal set of separators for a fat point is a good set of separators if the separators can be used to construct a basis for the vector space  $(I_{Z'}/I_Z)_{\underline{t}}$  for all  $\underline{t} \in \mathbb{N}^2$ .

Recall that by Remark 2.2 we can assume that none of the points in  $\text{Supp}(Z)$  lie on the lines defined by  $x_0$  and  $y_0$ . That is,  $x_0$  and  $y_0$  are nonzero-divisors in the rings  $R/I_Z$  and  $R/I_{Z'}$ . So, if  $\bar{0} \neq \bar{F} \in (I_{Z'}/I_Z)$ , then  $\bar{0} \neq \overline{x_0^a y_0^b F} \in (I_{Z'}/I_Z)$  for any  $(a, b) \in \mathbb{N}^2$ . With these observations in hand, we introduce the following definition.

**Definition 4.1.** Let  $Z$  and  $Z'$  be as in Convention 3.1, and let  $\{F_1, \dots, F_p\}$  be a set of minimal separators of the point  $P_i$  of multiplicity  $m_i$ . Let  $\deg F_i = (d_{i1}, d_{i2})$ . We call  $\{F_1, \dots, F_p\}$  a **good set of minimal separators** if for each  $\underline{t} = (t_1, t_2) \in \mathbb{N}^2$  the set

$$\left\{ \overline{x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1}, \dots, \overline{x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p} \right\}$$

is a linearly independent set of elements in  $(I_{Z'}/I_Z)_{\underline{t}}$ , where if  $t_j - d_{kj} < 0$  for some  $k$ , then the term  $\overline{x_0^{t_1-d_{k1}} y_0^{t_2-d_{k2}} F_k}$  is omitted.

**Example 4.2.** Consider the points  $P_1 = [1 : 0] \times [1 : 0]$  and  $P_2 = [1 : 1] \times [1 : 1]$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and set  $Z = \{P_1, P_2\}$  and  $Z' = \{P_1\}$ . Thus,  $I_Z = (x_1, y_1) \cap (x_1 - x_0, y_1 - y_0)$  and  $I_{Z'} = (x_1, y_1)$ . So  $(I_{Z'}/I_Z) = (\bar{x}_1, \bar{y}_1)$ . Now,  $\overline{y_0 x_1}$  and  $\overline{x_0 y_1}$  are both separators of  $P_2$  of degree  $\underline{t} = (1, 1)$  in  $(I_{Z'}/I_Z)_{\underline{t}}$ . However,  $\overline{y_0 x_1} - \overline{x_0 y_1} = \bar{0} \in (I_{Z'}/I_Z)_{\underline{t}}$  because  $y_0 x_1 - x_0 y_1 = y_0(x_1 - x_0) - x_0(y_1 - y_0) \in I_Z$ , so  $\overline{y_0 x_1}$  and  $\overline{x_0 y_1}$  are not linearly independent. Thus,  $\{\bar{x}_1, \bar{y}_1\}$  is not a good set of minimal separators.

A good set of minimal separators has the following useful properties.

**Theorem 4.3.** Let  $Z, Z'$  be as in Convention 3.1. Suppose that  $\{F_1, \dots, F_p\}$  is a good set of minimal separators of the point  $P_i$  of multiplicity  $m_i$ . Then

(i) for every  $\underline{t} \in \mathbb{N}^2$  a basis for  $(I_{Z'}/I_Z)_{\underline{t}}$  is given by

$$\left\{ \overline{x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1}, \dots, \overline{x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p} \right\};$$

(ii)  $\dim_k (I_{Z'}/I_Z)_{\underline{t}} = |\{F_i \mid \deg F_i \preceq \underline{t}\}|$  for all  $\underline{t} \succeq \underline{0}$ ; and

(iii)  $p = \deg Z - \deg Z' = \binom{m_i+N-1}{m_i-1} - \binom{m_i+N-2}{m_i-2}$ , where  $N = n + m$ .

*Proof.* Assume that  $P = P_i = [1 : 0 : \cdots : 0] \times [1 : 0 : \cdots : 0]$  so that  $I_P = (x_1, \dots, x_n, y_1, \dots, y_m)$ .

(i) By definition, the elements  $\{\overline{x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1}, \dots, \overline{x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p}\}$  form a linearly independent set in  $(I_{Z'}/I_Z)_{\underline{t}}$ , so it suffices to show that they also span  $(I_{Z'}/I_Z)_{\underline{t}}$ . For any  $\overline{H} \in (I_{Z'}/I_Z)_{\underline{t}}$ , there must exist homogeneous forms  $G_1, \dots, G_p$  such that

$$\overline{H} = \overline{G_1 F_1 + \cdots + G_p F_p} \text{ with } \deg G_i = \underline{t} - \deg F_i.$$

Rewrite each  $G_i$  as  $G_i = c_i x_0^{t_1-d_{i1}} y_0^{t_2-d_{i2}} + G'_i$  with  $G'_i \in I_P$ . For each  $i = 1, \dots, p$ , we have  $G'_i F_i \in I_Z$ . To see this, note that  $F_i \in I_{P_j}^{m_j}$  if  $P_j \neq P$ . On the other hand,  $F_i \in I_P^{m_i-1}$  and  $G'_i \in I_P$ , so  $G'_i F_i \in I_P^{m_i}$ . Hence,  $G'_i F_i \in I_Z = I_{P_1}^{m_1} \cap \cdots \cap I_P^{m_i} \cap \cdots \cap I_{P_s}^{m_s}$ . This implies

$$\overline{H} = \overline{c_1 x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1 + \cdots + c_p x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p},$$

i.e.,  $\overline{H}$  is in the span of  $\{\overline{x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1}, \dots, \overline{x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p}\}$ .

(ii) This follows directly from (i).

(iii) The second equality can be computed directly from the degree formula. We prove the first equality. For all  $\underline{t} \in \mathbb{N}^2$  we have a short exact sequence of vector spaces

$$(4.1) \quad 0 \longrightarrow (I_{Z'}/I_Z)_{\underline{t}} \longrightarrow (R/I_Z)_{\underline{t}} \longrightarrow (R/I_{Z'})_{\underline{t}} \longrightarrow 0.$$

Take any  $\underline{t} = (t_1, t_2) \gg \underline{0}$ , i.e.,  $t_i \gg 0$  for  $i = 1, 2$ . For any set of fat points  $Z$ , it is known that  $\dim_k(R/I_Z)_{\underline{t}} = \deg Z$  for  $\underline{t} \gg \underline{0}$  (cf. [17, Proposition 4.4]). Hence, if  $\underline{t} \gg \underline{0}$

$$\dim_k(I_{Z'}/I_Z)_{\underline{t}} = \dim_k(R/I_Z)_{\underline{t}} - \dim_k(R/I_{Z'})_{\underline{t}} = \deg Z - \deg Z'.$$

But by part (i), for  $\underline{t} \gg \underline{0}$ ,  $\dim_k(I_{Z'}/I_Z)_{\underline{t}} = p$ , so the conclusion follows.  $\square$

Recall that the **Hilbert function** of  $R/I_Z$  is the function  $H_Z : \mathbb{N}^2 \rightarrow \mathbb{N}$  defined by

$$H_Z(\underline{t}) := \dim_k(R/I_Z)_{\underline{t}} = \dim_k R_{\underline{t}} - \dim_k(I_Z)_{\underline{t}} \text{ for all } \underline{t} \in \mathbb{N}^2.$$

The Hilbert functions of  $Z$  and  $Z'$  are then linked by  $\deg_Z(P)$  when the minimal separators of  $P$  of multiplicity  $m$  are also a good set of minimal separators. The result follows directly from Theorem 4.3 (ii) and the short exact sequence (4.1).

**Corollary 4.4.** *Let  $Z$  and  $Z'$  be as in Convention 3.1, and suppose that  $\deg_Z(P) = (\underline{d}_1, \dots, \underline{d}_p)$  and that  $\{F_1, \dots, F_p\}$  is a good set of minimal separators. Then*

$$H_{Z'}(\underline{t}) = H_Z(\underline{t}) - |\{\underline{d}_j \mid \underline{d}_j \leq \underline{t}\}| \text{ for all } \underline{t} \in \mathbb{N}^2.$$

## 5. EXISTENCE OF GOOD SEPARATORS IN $\mathbb{P}^n \times \mathbb{P}^m$

As Theorem 4.3 suggests, a good set of minimal separators has some useful properties. A re-examination of the proof of [9, Theorem 3.3] shows that when  $Z$  is a set of fat points in  $\mathbb{P}^n$ , then the minimal separators of the point  $P_i$  of multiplicity  $m_i$  do form a good set of minimal separators. Further examination of this proof reveals that we need the fact that  $Z$  is ACM. We now show that if  $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$  is ACM, then for every point  $P \in \text{Supp}(Z)$ , the set of minimal separators of  $P$  forms a good set of minimal separators.

**Theorem 5.1.** *Suppose that  $Z = m_1 P_1 + \cdots + m_s P_s$  is a set of fat points in  $\mathbb{P}^n \times \mathbb{P}^m$ , and furthermore, suppose that  $Z$  is ACM. If  $\{F_1, \dots, F_p\}$  is a set of minimal separators of the point  $P_i$  of multiplicity  $m_i$ , then  $\{F_1, \dots, F_p\}$  is also a good set of minimal separators.*

*Proof.* After a change of coordinates, we can assume that  $P := P_i = [1 : 0 : \cdots : 0] \times [1 : 0 : \cdots : 0]$  and that  $\{x_0, y_0\}$  forms a maximal regular sequence (see Remark 2.2).

For each  $\underline{t} = (t_1, t_2) \in \mathbb{N}^2$ , we wish to show that the set

$$\left\{ \overline{x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1}, \dots, \overline{x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p} \right\}$$

is a linearly independent set in  $(I_{Z'}/I_Z)_{\underline{t}}$ . We can assume that  $t_1 - d_{j1} \geq 0$  and  $t_2 - d_{j2} \geq 0$  for all  $j = 1, \dots, p$ . If  $t_i - d_{ji} < 0$  for some  $j$ , we simply omit the term involving  $F_j$ .

Suppose, for a contradiction, that there exist nonzero constants  $c_1, \dots, c_p$  such that

$$\overline{c_1 x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1} + \cdots + \overline{c_p x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p} = \overline{0} \in (I_{Z'}/I_Z)_{\underline{t}},$$

or equivalently,

$$c_1 x_0^{t_1-d_{11}} y_0^{t_2-d_{12}} F_1 + \cdots + c_p x_0^{t_1-d_{p1}} y_0^{t_2-d_{p2}} F_p \in I_Z.$$

We can reorder the  $F_i$ 's so that  $0 \leq t_1 - d_{11} \leq t_1 - d_{21} \leq \cdots \leq t_1 - d_{p1}$ , and we factor out the largest possible power of  $x_0$ , i.e.,

$$x_0^{t_1-d_{11}} (c_1 y_0^{t_2-d_{12}} F_1 + \cdots + c_p x_0^{d_{11}-d_{p1}} y_0^{t_2-d_{p2}} F_p) \in I_Z.$$

Because  $Z$  is ACM and  $\overline{x_0}$  is a nonzero-divisor on  $R/I_Z$ , we get

$$(c_1 y_0^{t_2-d_{12}} F_1 + \cdots + c_e y_0^{t_2-d_{e2}} F_e + c_{e+1} x_0^{d_{11}-d_{e+1,1}} y_0^{t_2-d_{e+1,2}} F_{e+1} + \cdots + c_p x_0^{d_{11}-d_{p1}} y_0^{t_2-d_{p2}} F_p) \in I_Z.$$

Note, in the above expression, we are assuming that  $t_1 - d_{11} = \cdots = t_1 - d_{e1} < t_1 - d_{e+1,1}$ . The above expression thus implies that

$$(c_1 y_0^{t_2-d_{12}} F_1 + \cdots + c_e y_0^{t_2-d_{e2}} F_e) \in (I_Z, x_0).$$

We now factor out the largest possible  $y_0$  in the above polynomial. We relabel if necessary so that  $t_2 - d_{12} \leq t_2 - d_{i2}$  for  $i = 2, \dots, e$ . So, we get

$$y_0^{t_2-d_{12}} (c_1 F_1 + \cdots + c_e y_0^{d_{12}-d_{e2}} F_e) \in (I_Z, x_0).$$

Because  $\{x_0, y_0\}$  form a regular sequence on  $R/I_Z$ , we have that  $\overline{y_0}$  is a nonzero-divisor on  $R/(I_Z, x_0)$ . Thus, the previous expression implies that

$$(5.1) \quad (c_1 F_1 + \cdots + c_e y_0^{d_{12}-d_{e2}} F_e) \in (I_Z, x_0) \Leftrightarrow c_1 F_1 + \cdots + c_e y_0^{d_{12}-d_{e2}} F_e = H_1 + H_2 x_0$$

with  $H_1 \in I_Z$  and  $H_2 \in R$ . Note that if we rearrange the last expression, we get

$$H_2 x_0 = (c_1 F_1 + \cdots + c_e y_0^{d_{12}-d_{e2}} F_e) - H_1.$$

Since  $H_1 \in I_Z \subseteq I_{Z'}$  and  $F_1, \dots, F_e \in I_{Z'}$ , we get  $H_2 x_0 \in I_{Z'}$ . But  $x_0$  is a nonzero-divisor on  $R/I_{Z'}$ , so  $H_2 \in I_{Z'}$ .

So,  $H_2 \in I_Z$  or  $H_2 \in I_{Z'} \setminus I_Z$  since  $I_{Z'} = (I_{Z'} \setminus I_Z) \cup I_Z$ . However, if  $H_2 \in I_Z$ , then this would mean that

$$c_1 F_1 \in (I_Z, \hat{F}_1, F_2, \dots, F_p) \Leftrightarrow (\overline{F}_1, \dots, \overline{F}_p) = (\overline{F}_2, \dots, \overline{F}_p)$$

which contradicts the fact that the  $F_i$ 's are a minimal set of separators.

So, suppose  $H_2 \in I_{Z'} \setminus I_Z$ , or equivalently,  $\overline{H}_2 \neq \overline{0}$  in  $(I_{Z'}/I_Z)$ . Thus,

$$\overline{H}_2 = \overline{G_1 F_1} + \cdots + \overline{G_p F_p}$$

for some  $G_1, \dots, G_p$ . But by degree considerations,  $\deg F_1 \succ \deg H_2$ , so  $G_1 = 0$ . Hence

$$(5.2) \quad H_2 = G_2 F_2 + \cdots + G_p F_p + L \text{ with } L \in I_Z.$$

If we substitute (5.2) into (5.1), then we get

$$c_1 F_1 + \cdots + c_e y_0^{d_{12}-d_{e2}} F_e = H_1 + [G_2 F_2 + \cdots + G_p F_p + L]x_0$$

which, after rearranging and regrouping, gives

$$c_1 F_1 = K + K_2 F_2 + \cdots + K_p F_p \text{ with } K \in I_Z \text{ and } K_i \in R.$$

But this means that  $\overline{F}_1 \in (\overline{F}_2, \dots, \overline{F}_p) \subseteq R/I_Z$ , which again contradicts the fact that the  $F_i$ 's are a minimal set of separators. The conclusion now follows.  $\square$

In Example 3.9 we noted that  $|\deg_Z(P_2)| \neq \deg Z - \deg Z'$ , and that  $Z$  was not ACM. This can now be deduced from the next corollary.

**Corollary 5.2.** *Let  $Z$  and  $Z'$  be as in Convention 3.1. Suppose that there exists a point  $P$  in  $Z$  such that  $|\deg_Z(P)| \neq \deg Z - \deg Z'$ . Then  $Z$  is not ACM.*

*Proof.* If  $Z$  is ACM, then by the previous theorem, every point has a good set of minimal separators, whence  $|\deg_Z(P)| = \deg Z - \deg Z'$  by Theorem 4.3.  $\square$

**Example 5.3.** We compute the Hilbert function of  $Z = 3P$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Note that  $Z$  is ACM in  $\mathbb{P}^1 \times \mathbb{P}^1$ , so by Theorem 5.1 and Corollary 4.4, we get

$$H_{3P}(i, j) = H_{2P}(i, j) + |\{\underline{d}_l \in \deg_{3P}(P) \mid \underline{d}_l \preceq (i, j)\}|$$

and  $H_{2P} = H_P(i, j) + |\{\underline{d}_l \in \deg_{2P}(P) \mid \underline{d}_l \preceq (i, j)\}|$ . By Example 3.8,  $\deg_{3P}(P) = ((0, 2), (1, 1), (2, 0))$ , and  $\deg_{2P}(P) = ((0, 1), (1, 0))$ . Since  $H_P(i, j) = 1$  for all  $(i, j) \in \mathbb{N}^2$ ,

$$H_{3P} = \begin{bmatrix} 1 & 2 & 3 & 3 & \cdots \\ 2 & 4 & 5 & 5 & \cdots \\ 3 & 5 & 6 & 6 & \cdots \\ 3 & 5 & 6 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where position  $(i, j)$  of the matrix corresponds to  $H_{3P}(i, j)$  (the indexing starts at zero, not one). We can use this procedure to compute  $H_{mP}$  for any fat point  $mP \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ .

**Remark 5.4.** In a forthcoming paper [13], the authors give a formula for the degree of a separator of any fat point of an ACM fat point scheme  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  that requires only numerical information describing  $Z$ .



## 6. THE DEGREE OF A SEPARATOR AND THE MINIMAL RESOLUTION

In this section, we describe how  $\deg_Z(P_i)$  is encoded into the bigraded minimal free resolution of  $I_Z$  under certain hypotheses. Our results can be seen as a natural generalization of the case for reduced points in  $\mathbb{P}^n$  (see [1, 2]), reduced points in  $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$  (see [12]), and fat points in  $\mathbb{P}^n$  (see [9]).

We start with two technical lemmas that shall be required for our induction step.

**Lemma 6.1.** *Let  $Z$  and  $Z'$  be as in Convention 3.1. If  $\{F_1, \dots, F_p\}$  is a good set of minimal separators of  $P_i$  of multiplicity  $m_i$ , then*

$$(I_Z, F_1, \dots, F_{j-1}) : (F_j) = I_{P_i} \text{ for } j = 1, \dots, p.$$

*Proof.* We set  $\underline{d}_j := \deg F_j$  for  $j = 1, \dots, p$ .

To prove the inclusion  $I_{P_i} \subseteq (I_Z, F_1, \dots, F_{j-1}) : (F_j)$ , note that  $F_j \in I_{P_q}^{m_q}$  for all  $q \neq i$ , and for  $q = i$ ,  $F_j I_{P_i} \subseteq I_{P_i}^{m_i-1}$  since  $F_j \in I_{P_i}^{m_i-1}$ . Hence  $F_j I_{P_i} \subseteq I_Z \subseteq (I_Z, F_1, \dots, F_{j-1})$ .

Set  $P := P_i$ . To prove the other inclusion, we do a change of coordinates so that  $\bar{x}_0, \bar{y}_0$  are nonzero-divisors on  $R/I_Z$  and  $P = [1 : 0 : \cdots : 0] \times [1 : 0 : \cdots : 0]$ . Note that this means that  $I_P = (x_1, \dots, x_n, y_1, \dots, y_m)$ . Suppose that  $G \in (I_Z, F_1, \dots, F_{j-1}) : (F_j)$ , i.e.,  $GF_j \in (I_Z, F_1, \dots, F_{j-1})$ . Then there exist forms  $A_1, \dots, A_{j-1} \in R$  and  $A \in I_Z$  such that

$$(6.1) \quad GF_j = A + A_1 F_1 + \cdots + A_{j-1} F_{j-1} \Leftrightarrow GF_j - (A_1 F_1 + \cdots + A_{j-1} F_{j-1}) = A \in I_Z.$$

We can take  $G, A_1, \dots, A_{j-1}$  to be bihomogeneous. Furthermore, if  $\deg A = \underline{d} = (d_1, d_2)$ , then  $\deg G = \underline{d} - \underline{d}_j$  and  $\deg A_l = \underline{d} - \underline{d}_l$  for  $l = 1, \dots, j-1$ . We also write

$$G = c \underline{x}_0^{\underline{d}-\underline{d}_j} + G' \text{ and } A_l = a_l \underline{x}_0^{\underline{d}-\underline{d}_l} + A'_l$$

where we set  $\underline{x}_0^{\underline{b}} = x_0^{b_1} y_0^{b_2}$  with  $\underline{b} = (b_1, b_2)$ , and  $G', A'_1, \dots, A'_{j-1} \in I_P$ . Note that if for some  $k \in \{1, \dots, j-1\}$ , we have  $\underline{d} - \underline{d}_k \not\geq \underline{0}$ , then the term  $A_k F_k$  does not appear. Our goal is to show that  $c = 0$ , whence  $G = G' \in I_P$ .

It follows that  $G' F_j \in I_P^{m_i}$ , and similarly  $A'_l F_l \in I_P^{m_i}$  for  $l = 1, \dots, j-1$ . Because  $F_1, \dots, F_j \in I_{P_j}^{m_j}$  for  $j \neq i$ , we get

$$G' F_j - (A'_1 F_1 + \cdots + A'_{j-1} F_{j-1}) \in I_Z.$$

If we subtract this expression from (6.1), we get

$$c \underline{x}_0^{\underline{d}-\underline{d}_j} F_j - (a_1 \underline{x}_0^{\underline{d}-\underline{d}_1} F_1 + \cdots + a_{j-1} \underline{x}_0^{\underline{d}-\underline{d}_{j-1}} F_{j-1}) \in I_Z.$$

But then in  $(I_{Z'}/I_Z)_{\underline{d}}$  we have

$$(6.2) \quad \overline{c \underline{x}_0^{\underline{d}-\underline{d}_j} F_j - (a_1 \underline{x}_0^{\underline{d}-\underline{d}_1} F_1 + \cdots + a_{j-1} \underline{x}_0^{\underline{d}-\underline{d}_{j-1}} F_{j-1})} = \bar{0}.$$

Since the separators  $F_1, \dots, F_p$  are a good set of minimal separators, the elements

$$\left\{ \overline{\underline{x}_0^{\underline{d}-\underline{d}_1} F_1}, \dots, \overline{\underline{x}_0^{\underline{d}-\underline{d}_j} F_j} \right\}$$

are linearly independent in  $(I_{Z'}/I_Z)_{\underline{d}}$ . Thus equation (6.2) holds only if  $c = 0$ . But this means that  $G = G' \in I_P$ , as desired.  $\square$

We need the following result from homological algebra (see [21, Exercise 4.1.2]); here, we use  $\text{pdim}(N)$  to denote the **projective dimension** of an  $R$ -module  $N$ .

**Lemma 6.2.** *Let  $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$  be a short exact sequence of  $R$ -modules. If  $\text{pdim}(M'') \neq \text{pdim}(M) + 1$ , then  $\text{pdim}(M') = \max\{\text{pdim}(M), \text{pdim}(M'')\}$ .*

**Lemma 6.3.** *Let  $Z, Z'$  be as in Convention 3.1, and suppose that  $\{F_1, \dots, F_p\}$  is a good set of minimal separators of the point  $P_i$  of multiplicity  $m_i$ . If  $Z'$  is ACM, then  $\text{pdim}(R/(I_Z, F_1, \dots, F_j)) = N = n + m$  for  $j = 1, \dots, p$ .*

*Proof.* For each  $j = 1, \dots, p$ , we have the short exact sequence  
(6.3)

$$0 \rightarrow (R/(I_Z, F_1, \dots, F_{j-1}) : (F_j))(-\underline{d}_j) \xrightarrow{\times F_j} R/(I_Z, F_1, \dots, F_{j-1}) \rightarrow R/(I_Z, F_1, \dots, F_j) \rightarrow 0.$$

where  $\underline{d}_j = \deg F_j$ . But we know from Lemma 6.1 that  $(I_Z, F_1, \dots, F_{j-1}) : (F_j) = I_{P_i}$ . So, the short exact sequence (6.3) becomes

$$(6.4) \quad 0 \longrightarrow (R/I_{P_i})(-\underline{d}_j) \xrightarrow{\times F_j} R/(I_Z, F_1, \dots, F_{j-1}) \longrightarrow R/(I_Z, F_1, \dots, F_j) \longrightarrow 0.$$

By Lemma 2.3, we have  $\text{pdim}(R/I_P) = N$  where  $N = n + m$ . We now do descending induction on  $j$ . When  $j = p$ , then  $I_{Z'} = (I_Z, F_1, \dots, F_p)$ , and  $R/I_{Z'}$  is CM by hypothesis. Since  $\dim R/I_Z = 2$ , we have  $\text{pdim}(R/I_{Z'}) = N$ . For  $j = p$ , the exact sequence (6.4) becomes:

$$0 \longrightarrow (R/I_{P_i})(-\underline{d}_p) \xrightarrow{\times F_p} R/(I_Z, F_1, \dots, F_{p-1}) \longrightarrow R/(I_Z, F_1, \dots, F_p) \longrightarrow 0.$$

Because  $\text{pdim}(R/I_{P_i}) = \text{pdim}(R/(I_Z, F_1, \dots, F_p)) = N$ , Lemma 6.2 implies

$$\text{pdim } R/(I_Z, F_1, \dots, F_{p-1}) = \max\{\text{pdim}(R/I_{P_i}), \text{pdim}(R/(I_Z, F_1, \dots, F_p))\} = N.$$

For  $j \leq p - 1$ , we apply the induction hypothesis to (6.4) and again use Lemma 6.2.  $\square$

We come to the main result of this section which states that under certain hypotheses, the entries of  $\deg_Z(P_i)$  are encoded into the minimal free resolution of  $I_Z$ .

**Theorem 6.4.** *Let  $Z, Z'$  be sets of fat points as in Convention 3.1. Suppose that  $Z$  is ACM, so that the minimal  $\mathbb{N}^2$ -graded free resolution of  $R/I_Z$  has the form*

$$0 \rightarrow \mathbb{F}_N \rightarrow \dots \rightarrow \mathbb{F}_1 \rightarrow R \rightarrow R/I_Z \rightarrow 0$$

where  $N = n + m$ . If  $Z'$  is ACM, then

$$\mathbb{F}_N = R(-\underline{d}_1 - \underline{N}) \oplus \dots \oplus R(-\underline{d}_p - \underline{N}) \oplus \mathbb{F}'_N$$

where  $\deg_Z(P_i) = (\underline{d}_1, \dots, \underline{d}_p)$  and  $\underline{N} = (n, m)$ .

*Proof.* Let  $\mathcal{H}_0$  denote the minimal free resolution of  $I_Z$  and let  $F_1, \dots, F_p$  be a set of minimal separators. We order them with respect to the lexicographical ordering, i.e.,  $\deg F_1 = \underline{d}_1 \leq \dots \leq \deg F_p = \underline{d}_p$ . Since  $Z$  is ACM, the set  $F_1, \dots, F_p$  is also a good set of minimal separators by Theorem 5.1. We will add each  $F_1, \dots, F_p$  to  $I_Z$  one at a time, and then consider the resolution of  $(I_Z, F_1, \dots, F_j)$  for  $j = 1, \dots, p$ .

When  $j = 1$ , we have the short exact sequence

$$(6.5) \quad 0 \rightarrow R/((I_Z) : (F_1))(-\underline{d}_1) = (R/I_{P_i})(-\underline{d}_1) \xrightarrow{\times F_1} R/I_Z \rightarrow R/(I_Z, F_1) \rightarrow 0.$$

By Lemma 2.3, the resolution of  $R/I_{P_i}$  has form

$$0 \rightarrow \mathbb{G}_N = R(-\underline{N}) \rightarrow \mathbb{G}_{N-1} \rightarrow \cdots \rightarrow \mathbb{G}_1 \rightarrow R \rightarrow R/I_{P_i} \rightarrow 0$$

where  $N = n + m$ . Applying the mapping cone construction to (6.5) we get a resolution of  $I_1 = (I_Z, F_1)$ :

$$(6.6) \quad \mathcal{H}_1 : 0 \rightarrow R(-\underline{d}_1 - \underline{N}) \rightarrow \mathbb{F}_N \oplus \mathbb{G}_{N-1}(-\underline{d}_1) \rightarrow \cdots \rightarrow \mathbb{F}_1 \oplus R(-\underline{d}_1) \rightarrow R \rightarrow R/I_1 \rightarrow 0$$

where  $\underline{d}_1 = (d_{11}, d_{12})$  and  $\underline{N} = (n, m)$ .

The resolution of  $I_1$  given in (6.6) is too long since  $\text{pdim}(R/I_1) = N$  by Lemma 6.3. Thus,  $R(-\underline{d}_1 - \underline{N})$  must be part of the trivial complex  $\mathcal{T}$ , and to obtain a minimal resolution, the term  $R(-\underline{d}_1 - \underline{N})$  must cancel with something in

$$\mathbb{F}_N \oplus \mathbb{G}_{N-1}(-\underline{d}_1) = \mathbb{F}_N \oplus R^n(-d_{11} - n + 1, -d_{12} - m) \oplus R^m(-d_{11} - n, -d_{12} - m + 1).$$

By degree considerations, we cannot cancel the term  $R(-\underline{d}_1 - \underline{N})$  with any of the terms of  $R^n(-d_{11} - n + 1, -d_{12} - m) \oplus R^m(-d_{11} - n, -d_{12} - m + 1)$ . Thus,  $\mathbb{F}_N = \mathbb{F}'_N \oplus R(-\underline{d}_1 - \underline{N})$ , i.e., the term  $R(-\underline{d}_1 - \underline{N})$  must cancel with something in  $\mathbb{F}_N$ . Note that after we cancel  $R(-\underline{d}_1 - \underline{N})$ , we get a resolution of  $I_1$  which may or may not be minimal. We let

$$\mathcal{H}_1 : 0 \rightarrow \mathbb{F}'_N \oplus \mathbb{G}_{N-1}(-\underline{d}_1) \rightarrow \cdots \rightarrow R \rightarrow R/I_1 \rightarrow 0$$

denote this resolution; we shall require this resolution at the induction step.

More generally, for our induction step, assume that we have shown that a resolution of  $I_{j-1} = (I_Z, F_1, \dots, F_{j-1})$  is given by

$$\mathcal{H}_{j-1} : 0 \rightarrow \mathbb{F}'_N \oplus \mathbb{G}_{N-1}(-\underline{d}_1) \oplus \cdots \oplus \mathbb{G}_{N-1}(-\underline{d}_{j-1}) \rightarrow \cdots \rightarrow R \rightarrow R/I_{j-1} \rightarrow 0$$

and that  $\mathbb{F}_N = R(-\underline{d}_1 - \underline{N}) \oplus \cdots \oplus R(-\underline{d}_{j-1} - \underline{N}) \oplus \mathbb{F}'_N$ . We have a short exact sequence

$$(6.7) \quad 0 \rightarrow R/((I_{j-1} : (F_j))(-\underline{d}_j)) \xrightarrow{\times F_j} R/I_{j-1} \rightarrow R/I_j \rightarrow 0$$

where  $I_j = (I_Z, F_1, \dots, F_j)$ .

We apply the mapping cone construction to (6.7) along with the resolution  $\mathcal{H}_{j-1}$  to make a resolution of  $R/I_j$ . Since  $R/((I_{j-1} : (F_j))(-\underline{d}_j)) \cong R/I_{P_i}(-\underline{d}_j)$ , the mapping cone produces the resolution:

$$\mathcal{K}_j : 0 \rightarrow R(-\underline{d}_j - \underline{N}) \rightarrow \mathbb{F}'_N \oplus \mathbb{G}_{N-1}(-\underline{d}_1) \oplus \cdots \oplus \mathbb{G}_{N-1}(-\underline{d}_j) \rightarrow \cdots \rightarrow R \rightarrow R/I_j \rightarrow 0.$$

This resolution is too long by Lemma 6.3, so  $R(-\underline{d}_j - \underline{N})$  must cancel with a term in

$$\mathbb{F}'_N \oplus \mathbb{G}_{N-1}(-\underline{d}_1) \oplus \cdots \oplus \mathbb{G}_{N-1}(-\underline{d}_j).$$

The term  $R(-\underline{d}_j - \underline{N})$  cannot cancel with any term in  $\mathbb{G}_{N-1}(-\underline{d}_j)$  by degree considerations. So, suppose that  $R(-\underline{d}_j - \underline{N})$  cancels with some term in

$$\mathbb{G}_{N-1}(-\underline{d}_l) = R^n(-d_{l1} - n + 1, -d_{l2} - m) \oplus R^m(-d_{l1} - n, -d_{l2} - m + 1)$$

for some  $1 \leq l < j$ . Hence, either

$$(-d_{j1} - n, -d_{j2} - m) = (-d_{l1} - n, -d_{l2} - m + 1)$$

from which we get  $d_{j1} = d_{l1}$ , and  $d_{j2} = d_{l2} - 1$ . But this is not possible since we have ordered  $\underline{d}_1 \leq \dots \leq \underline{d}_p$  with respect to the lexicographical ordering. Or

$$(-d_{j1} - n, -d_{j2} - m) = (-d_{l1} - n + 1, -d_{l2} - m)$$

from which we get  $d_{j1} = d_{l1} - 1$ , and  $d_{j2} = d_{l2}$ . But again this is not possible because of the ordering of  $\underline{d}_1 \leq \dots \leq \underline{d}_p$ .

Hence, the term  $R(-\underline{d}_j - \underline{N})$  must cancel with some term in  $\mathbb{F}'_N$ . Hence,  $\mathbb{F}'_N = \mathbb{F}''_N \oplus R(-\underline{d}_j - \underline{N})$ . The result now follows by induction on  $j$ .  $\square$

As a corollary, we can bound on the rank of the last syzygy module.

**Corollary 6.5.** *With the hypotheses as in Theorem 6.4 (i), let  $M = \max\{m_1, \dots, m_s\}$  and  $N = n + m$ . Then*

$$\text{rk } \mathbb{F}_N \geq \binom{M + N - 2}{N - 1}.$$

*Proof.* Suppose  $P_i$  has multiplicity  $M$ . Then by Theorem 4.3 and Theorem 6.4, at least  $|\deg_Z(P)| = \deg Z - \deg Z' = \binom{M+N-2}{N-1}$  shifts appear in  $\mathbb{F}_N$ .  $\square$

## 7. FUTURE DIRECTIONS

All of the definitions and results in this paper, except Theorem 5.1, can be easily generalized to  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ . However, the existence of good sets of minimal separators, when  $r \geq 3$ , appears difficult to prove. We propose the following question:

**Question 7.1.** *Suppose that  $Z = m_1 P_1 + \dots + m_s P_s$  is a set of ACM fat points in  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ . Is it true that the set of minimal separators for any fat point of  $Z$  is a good set of minimal separators?*

In the proof of Theorem 5.1, we used equation (5.1) and the fact that  $x_0$  is a nonzero-divisor to show that  $H_2 \in I_{Z'}$ , from which we derive our contradiction. In trying to generalize our proof to the case  $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$  with  $r \geq 3$ , we end up with an expression similar to (5.1), but involving more nonzero-divisors. For example, when  $r = 3$ , (and using the variables  $x_i, y_i$  and  $z_i$ ) we can show that there exists an element of the form  $H_1 + H_2 x_0 + H_3 y_0$  with  $H_1 \in I_Z$  and  $H_2, H_3 \in R$ , and that this element is some combination of the separators. Thus, there is an element of the form  $H_2 x_0 + H_3 y_0 \in I_{Z'}$ , but unlike the bigraded case, we do not see how to use the fact that  $x_0$  is also a nonzero-divisor.

We end with some evidence for this question. Question 7.1 is true for  $r = 1$  (see proof of [9, Theorem 3.3]) and  $r = 2$ , as proved in this paper. Question 7.1 also holds if  $m_1 = \dots = m_s = 1$  for any  $r \geq 1$ . This result follows from [11, Theorem 5.7] where it is shown that  $|\deg_Z(P)| = 1$  when  $Z$  is ACM. In other words,  $(I_{Z'}/I_Z) = (\overline{F})$  is principally generated, and  $\left\{ \overline{x_0^{t-\deg F} F} \right\}$  is a linearly independent set in  $(I_{Z'}/I_Z)_t$  for all  $t$ .

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